

COHOMOLOGY OF ALGEBRAIC GROUPS, FINITE GROUPS, AND LIE ALGEBRAS: INTERACTIONS AND CONNECTIONS

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1. OVERVIEW

1.1. Let G be a simple, simply connected algebraic group scheme defined over finite field \mathbb{F}_p , and $F^r : G \rightarrow G$ be the r th iteration of the Frobenius morphism. Let G_r be the scheme theoretic kernel of F^r , and $G(\mathbb{F}_q)$ be the finite Chevalley group obtained by looking at the fixed points under F^r . For a given rational G -module M one can consider the restriction of the action of M to the infinitesimal Frobenius kernel G_r and to the finite group $G(\mathbb{F}_q)$.

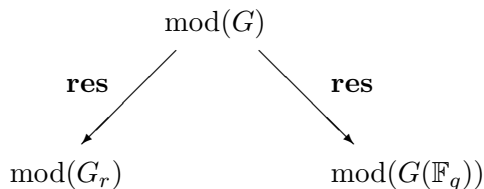


FIGURE 1.

For over 40 years, representation theorists have attempted to understand the relationship between these three module categories. Two basic questions we will address in this paper are:

(1.1.1) What is the relationship between the representation theories of these categories?

(1.1.2) Is there a relationship between the cohomology theories of these categories?

Even though there is no direct functorial relationship between $\text{mod}(G_r)$ and $\text{mod}(G(\mathbb{F}_q))$ these categories have very similar features. For example, both categories correspond to module categories for finite dimensional cocommutative Hopf algebras. Curtis proved in the 1960s there exists a one-to-one correspondence between the simple G_r -modules and simple $G(\mathbb{F}_q)$ -modules. This correspondence is given by simply restricting natural classes of simple G -modules. Furthermore, Steinberg's Tensor Product Theorem allows one to easily transfer the questions pertaining to the computation of irreducible characters between these three categories. The theme of lifting G_r -modules to G -structures was further studied by Ballard and Jantzen for projective G_r -modules. For a general guide to the history and results in this area we refer the reader to [Hu2].

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The aim of this paper is to present new methods in relating these three categories which have been developed over the past 10 years. Even though the group $G(\mathbb{F}_p)$ is finite, we will see that there are ambient geometric structures which govern its representation theory and cohomology. One theme in these new approaches has been to use nilpotent orbit theory for semisimple Lie algebras and the geometry of the flag variety G/B to calculate extensions of modules over the group $G(\mathbb{F}_p)$. Often times the use of such tools can reduce difficult group cohomology calculations to combinatorics within the given root system.

1.2. This paper is organized as follows. In Section 2, we review the basic facts about the representation theory of G , G_r and $G(\mathbb{F}_q)$. We give a parametrization of the simple modules for each of these cases and describe how the simple modules are related. The projective modules for G_r and $G(\mathbb{F}_q)$ are discussed in relation to proving the existence of G -structures. In the following section (Section 3), we give the fundamental definitions and properties of the complexity and support varieties of a module for a finite dimensional cocommutative Hopf algebra (i.e., finite group scheme). In Section 4, we discuss the relationships between the complexity and support varieties of a given finite dimensional G -module when restricted to $\text{mod}(G_1)$ and $\text{mod}(G(\mathbb{F}_p))$. These results were motivated by questions posed by Parshall in the 1980s. In Section 5, we investigate the connections between the cohomology groups in $\text{mod}(G)$, $\text{mod}(G_r)$ and $\text{mod}(G(\mathbb{F}_q))$ using techniques developed by Bendel, Pillen and the author. Formulas for Ext^1 -groups will be exhibited, in addition to results pertaining to the existence of self-extensions. Finally, in Section 6, we present new developments in the computation of the cohomology groups $H^\bullet(G(\mathbb{F}_q), k)$. In the case when $r = 1$ and p is larger than the Coxeter number, we give an upper bound for the dimension of these groups via Kostant's Partition Functions. Moreover, our methods allow us to give precise information about the existence of the first non-trivial cohomology when the root system is of type A or C .

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2. REPRESENTATION THEORY

2.1. Notation: Throughout this paper, we will follow the basic Lie theoretic conventions used in [Jan1]. Let G be a simple simply connected algebraic group scheme which is defined and split over the finite field \mathbb{F}_p with p elements. Moreover, let k denote an algebraically closed field of characteristic p . Let T be a maximal split torus and Φ be the root system associated to (G, T) . The positive (resp. negative) roots will be denoted by Φ^+ (resp. Φ^-), with Δ the collection of simple roots. Let B be a Borel subgroup containing T corresponding to the negative roots and U be the unipotent radical of B . For a given root system of rank n , the simple roots in Δ will be denoted by $\alpha_1, \alpha_2, \dots, \alpha_n$. Our convention will be to employ the Bourbaki ordering of simple roots. In particular, for type B_n , α_n denotes the unique short simple root and for type C_n , α_n denotes the unique long simple root. The highest short root is labelled by α_0 .

Let \mathbb{E} be the Euclidean space associated with Φ , and the inner product on \mathbb{E} will be denoted by $\langle \cdot, \cdot \rangle$. Set $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ (the coroot corresponding to $\alpha \in \Phi$). The fundamental weights (basis

dual to $\alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_n^\vee$ will be denoted by $\omega_1, \omega_2, \dots, \omega_n$. Let W denote the Weyl group associated to Φ and for $w \in W$, let $\ell(w)$ denote the length of the word.

Let $X(T)$ be the integral weight lattice spanned by the fundamental weights. For $\lambda, \mu \in X(T)$, $\lambda \geq \mu$ if and only if $\lambda - \mu$ is a positive integral linear combination of simple roots. The set of dominant integral weights is defined by

$$X(T)_+ := \{\lambda \in X(T) : 0 \leq \langle \lambda, \alpha^\vee \rangle, \text{ for all } \alpha \in \Delta\}.$$

For a weight $\lambda \in X(T)$, set $\lambda^* := -w_0\lambda$ where w_0 is the longest word in the Weyl group W . The dot action is defined by $w \cdot \lambda := w(\lambda + \rho) - \rho$ on $X(T)$ where ρ is the half-sum of the positive roots. For $\alpha \in \Delta$, $s_\alpha \in W$ denotes the reflection in the hyperplane determined by α . The Coxeter number h equals $\langle \rho, \alpha_0^\vee \rangle + 1$.

Let M be a T -module. Then $M = \oplus M_\lambda$ where M_λ are the weight spaces of M . The formal character of M is $\text{ch } M = \sum_{\lambda \in X(T)} \dim M_\lambda e(\lambda)$. An important class of G -modules whose formal character is known are the induced or Weyl modules. For $\lambda \in X(T)_+$, let

$$H^0(\lambda) := \text{ind}_B^G \lambda = \{f : G \rightarrow \lambda : f(g \cdot b) = \lambda(b^{-1})f(g), \text{ for all } b \in B, g \in G\}$$

be the induced module. These G -modules can be interpreted geometrically as the global sections of the line bundle $\mathcal{L}(\lambda)$ over the flag variety G/B . The Weyl module of highest weight λ is defined by $V(\lambda) := H^0(\lambda^*)^*$. The following theorem provides basic properties of the induced/Weyl modules.

Theorem 2.1.1. (a) $H^0(\lambda) \neq 0$ if and only if $\lambda \in X(T)_+$;
 (b) $\dim H^0(\lambda)_\lambda = 1$, $H^0(\lambda)_\mu \neq 0$ implies that $w_0\lambda \leq \mu \leq \lambda$;
 (c) $\dim H^0(\lambda) = \prod_{\alpha \in \Phi^+} \frac{\langle \lambda + \rho, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle}$.

If H is an algebraic group scheme defined over \mathbb{F}_p then there exists a Frobenius morphism $F : H \rightarrow H$. In the case when $H = GL_n(-)$, F takes a matrix (a_{ij}) to (a_{ij}^p) . Let F^r be the r -th iteration of the F with itself. For a rational H -module M , let $M^{(r)}$ be the module obtained by composing the underlying representation for M with F^r . Moreover, let M^* denote the dual module.

We will be interested in understanding the representation theory and cohomology of infinitesimal Frobenius kernels for reductive algebraic groups G . For $r \geq 1$, let $G_r := \ker F^r$ be the r th Frobenius kernel of G and $G(\mathbb{F}_q)$ be the associated finite Chevalley group obtained by looking at the \mathbb{F}_q rational points of G . We note that in the case when $r = 1$, $\text{Mod}(G_1)$ is equivalent to $\text{Mod}(u(\mathfrak{g}))$ where $\mathfrak{g} = \text{Lie } G$ and $u(\mathfrak{g})$ is the restricted enveloping algebra of \mathfrak{g} .

2.2. Simple Modules: In this section we will describe the classification of simple modules in $\text{mod}(G)$, $\text{mod}(G_r)$ and $\text{mod}(G(\mathbb{F}_q))$. All modules will be defined over the field k . The details can be found in [Jan1, II. Chapter 2-3].

First for $\lambda \in X(T)_+$, the socle of $H^0(\lambda)$ (largest semisimple submodule) is simple. Set $L(\lambda) = \text{soc}_G H^0(\lambda)$. Then the simple finite-dimensional modules are given by: $\{L(\lambda) : \lambda \in X(T)_+\}$. Note that when $k = \mathbb{C}$ then $L(\lambda) \cong H^0(\lambda)$ for all $\lambda \in X(T)_+$.

In order to classify the simple modules in $\text{mod}(G_r)$ and $\text{mod}(G(\mathbb{F}_q))$ we need to define the set of p^r -restricted weights. Let

$$X_r(T) = \{\lambda \in X(T)_+ : 0 \leq \langle \lambda, \alpha^\vee \rangle \leq p^r - 1, \text{ for all } \alpha \in \Delta\}.$$

If $\lambda \in X_r(T)$ then $L(\lambda)|_{G_r}$ remain simple, and

(2.2.1) $\{\text{simple finite-dimensional modules for } G_r\} \leftrightarrow \{L(\lambda) \downarrow_{G_r} : \lambda \in X_r(T)\}$ (one to one correspondence).

On the other hand, Curtis proved that one can restrict $L(\lambda)$ to $G(\mathbb{F}_q)$ and also get a classification of simple $G(\mathbb{F}_q)$ -modules:

(2.2.2) $\{\text{simple finite-dimensional modules for } G(\mathbb{F}_q)\} \leftrightarrow \{L(\lambda) \downarrow_{G(\mathbb{F}_q)} : \lambda \in X_r(T)\}$ (one to one correspondence).

Now by a beautiful theorem of Steinberg (known as Steinberg's Tensor Product Theorem), one can describe all simple G -modules by tensoring by Frobenius twists of simple G_1 -modules:

Theorem 2.2.1. *Let $\lambda \in X(T)_+$, $\lambda = \lambda_0 + \lambda_1 p + \cdots + \lambda_s p^s$, where $\lambda_i \in X_1(T)$. Then*

$$L(\lambda) \cong L(\lambda_0) \otimes L(\lambda_1)^{(1)} \otimes \cdots \otimes L(\lambda_s)^{(s)}.$$

If G is semisimple then $X_1(T)$ is finite. As a consequence of the Steinberg Tensor Product Theorem, if we know the characters of the simple G_1 -modules $L(\lambda)$ where $\lambda \in X_1(T)$, we can compute the characters of all simple finite-dimensional G -modules. The characters of the simple G -modules are still undetermined. For $p > h$ there is a conjecture due to Lusztig which give a recursive formula for the characters of simple modules via Kazhdan-Lusztig polynomials (cf. [Jan1, Chapter C]).

2.3. Projective Modules: For $r \geq 1$, the infinitesimal Frobenius kernel G_r is a finite group scheme whose module category is equivalent to $\text{Mod}(\text{Dist}(G_r))$ where $\text{Dist}(G_r)$ is a finite-dimensional cocommutative Hopf algebra. On the other hand, the category of $G(\mathbb{F}_q)$ modules that we are considering is equivalent to $\text{Mod}(kG(\mathbb{F}_q))$ where $kG(\mathbb{F}_q)$ is the group algebra of $G(\mathbb{F}_q)$ (also a finite-dimensional cocommutative Hopf algebra). In both instances the distribution algebra $\text{Dist}(G_r)$ and $kG(\mathbb{F}_q)$ are symmetric algebra. This means that these algebras are self-injective and the projective cover of a simple module is isomorphic to its injective hull.

If $L(\lambda)$ ($\lambda \in X_r(T)$) is a simple module for G_r then let $Q_r(\lambda)$ be its injective hull (and projective cover). One natural question to ask is whether we can lift the G_r -structure on $Q_r(\lambda)$ to a compatible G -module structure. Ballard showed that this holds when $p \geq 3(h-1)$, and Jantzen later lowered the bound to $p \geq 2(h-1)$.

Theorem 2.3.1. *Let $\lambda \in X_r(T)$ and $p \geq 2(h-1)$. Then $Q_r(\lambda)$ admits a G -structure.*

The idea behind the proof given in [Jan1, II 11.11] is to first realize $Q_r(\lambda)$ as a G_r -direct summand of the G -module $\text{St}_r \otimes L$, where L is an irreducible G -module and St_r is the r th Steinberg module. The proof entails showing that $Q_r(\lambda)$ is in fact a G -direct summand when $p \geq 2(h-1)$.

The Steinberg module St_r is projective upon restriction to G_r and $G(\mathbb{F}_q)$. Therefore, $\text{St}_r \otimes L$ is projective upon restriction to $G(\mathbb{F}_q)$. Consequently, for $p \geq 2(h-1)$, one can lift $Q_r(\lambda)$ to a G -structure and upon restriction to $G(\mathbb{F}_q)$ this module will split into a direct summand of projective indecomposable $kG(\mathbb{F}_q)$ -modules. For $\lambda \in X_r(T)$, let $U_r(\lambda)$ be the projective cover of $L(\lambda)$ in $\text{mod}(kG(\mathbb{F}_q))$. Chastkofsky [Ch] proved that

$$[Q_r(\lambda)|_{G(\mathbb{F}_q)} : U_r(\mu)] = \sum_{\nu \in \Gamma} [L(\mu) \otimes L(\nu) : L(\lambda) \otimes L(\nu)^{(r)}]_G.$$

Here $\Gamma = \{\nu \in X(T)_+ : \langle \nu, \alpha_0^\vee \rangle < h\}$. We shall see later that similar factors appear in Ext^1 -formulas relating the extensions of simple modules for $G(\mathbb{F}_q)$ to simple modules for G .

3. HOMOLOGICAL ALGEBRA

3.1. Complexity: Let $\mathcal{V} = \{V_j : j \in \mathbb{N}\}$ be a sequence of finite-dimensional k -vector spaces. The rate of growth of \mathcal{V} , denoted by $r(\mathcal{V})$, is the smallest positive integer c such that $\dim_k V_n \leq K \cdot n^{c-1}$ for some constant $K > 0$.

Given a finite-dimensional algebra A , Alperin [Al] introduced the notion of the complexity of a module in 1977 as a generalization of the notion of periodicity in group cohomology. The complexity is defined as follows. Let $M \in \text{mod}(A)$ and

$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

be a minimal projective resolution of M . This means that the exact sequence above is a projective resolution of M and the kernels have no projective summands. The complexity of M in $\text{mod}(A)$ is $r(\{P_t : t \in \mathbb{N}\})$.

Let $\text{Ext}_A^j(M, -)$ be the j th right derived functor of $\text{Hom}_A(M, -)$ for M in $\text{Mod}(A)$. If A is an augmented algebra (i.e., A admits the trivial module k), and N is in $\text{Mod}(A)$ set $H^j(A, N) = \text{Ext}_A^j(k, N)$. One can relate the complexity of N in terms of the rate of growth of extension groups.

Theorem 3.1.1. *Let A be a finite-dimensional algebra and S_1, S_2, \dots, S_u be a complete set of simple objects in $\text{mod}(A)$ then*

$$c_A(M) = r(\{\text{Ext}_A^t(\oplus_{j=1}^u S_j, N) : t \in \mathbb{N}\}).$$

The complexity becomes a finer invariant when one considers the case when A is self-injective, in particular when A is a finite-dimensional Hopf algebra. In this setting a module being injective is equivalent to it being projective, and the complexity is an invariant of projectivity.

Proposition 3.1.2. *Let A be a self injective algebra. Then $c_A(M) = 0$ if and only if M is projective.*

Proof. This can be shown as follows. Suppose $c_A(M) = 0$, then

$$0 \rightarrow P_s \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

is a minimal projective resolution of M . Furthermore, P_s is injective so this sequence splits, hence M is projective. On the other hand if M is projective one can construct a resolution of M , $0 \rightarrow M \rightarrow M \rightarrow 0$, thus $c_A(M) = 0$. \square

3.2. Support Varieties: Support varieties were introduced in the pioneering work of Carlson [Ca1] [Ca2] nearly 30 years ago as a method to study complexes and resolutions of modules over group algebras. Since that time the theory of support varieties has been extended to restricted Lie algebras by Friedlander and Parshall [FP1], to finite-dimensional sub Hopf algebras of the Steenrod algebra by Palmieri and the author [NP1], to infinitesimal group schemes by Suslin, Friedlander and Bendel [SFB1] [SFB2], and to arbitrary finite-dimensional cocommutative Hopf algebras by Friedlander and Pevtsova [FPe]. Further attempts to generalize the theory have been made to finite-dimensional algebras by Solberg and Snashall [SS] via Hochschild cohomology and Hecke algebras by Erdmann and Holloway [EH], Lie superalgebras [BKN1, BKN2, BKN3, BaKN], quantum

groups [Ost] [BNPP], and quantum complete intersections [BE]. As a testament to the dissemination of this theory, Dmitry Rumynin wrote in a Mathematical Review (MR: 2003b:20063) that “support varieties have proved to be an indispensable tool in the arsenal of a modern representation theorist.”

Friedlander and Suslin [FS] proved that for a finite-dimensional cocommutative Hopf algebra A , the even degree cohomology ring $R = H^{2\bullet}(A, k)$ is a commutative finitely generated algebra, and $H^\bullet(A, M)$ is a finitely generated R -module for any M in $\text{mod}(A)$. Set $\mathcal{V}_A(k) = \text{Maxspec}(R)$. For any finitely generated A -module M , one can assign a conical subvariety $\mathcal{V}_A(M)$ inside the spectrum of the cohomology ring $\mathcal{V}_A(k)$ by letting

$$\mathcal{V}_A(M) = \text{Maxspec}(R/J_M)$$

where $J_M = \text{Ann}_R \text{Ext}_A^\bullet(k, M \otimes M^*)$. Here M^* is the dual module of M .

Support varieties provide a method to introduce the geometry of the spectrum of R into the representation theory of A . These varieties have natural geometric properties and provide a geometric interpretation of the complexity of M :

$$(3.2.1) \quad c_A(M) = \dim \mathcal{V}_A(M)$$

$$(3.2.2) \quad \mathcal{V}_A(M \oplus N) = \mathcal{V}_A(M) \cup \mathcal{V}_A(N)$$

$$(3.3.3) \quad \mathcal{V}_A(M \otimes N) = \mathcal{V}_A(M) \cap \mathcal{V}_A(N)$$

$$(3.3.4) \quad \text{If } W \text{ is a conical subvariety of } \mathcal{V}_A(k) \text{ there exists } M \text{ in } \text{mod}(A) \text{ such that } \mathcal{V}_A(M) = W.$$

$$(3.3.5) \quad \text{If } M \in \text{mod}(A) \text{ is indecomposable then } \text{Proj}(\mathcal{V}_A(M)) \text{ is connected.}$$

For finite groups a description of the spectrum of the cohomology ring was determined by Quillen [Q1, Q2] using elementary abelian subgroups. Avrunin and Scott [AS] proved a more general stratification theory for $\mathcal{V}_A(M)$. Support varieties become very transparent when one considers the case when \mathfrak{g} is a restricted Lie algebra over a field k of characteristic $p > 0$ and A is the restricted universal enveloping algebra $u(\mathfrak{g})$. In this situation the spectrum of the cohomology ring is homeomorphic to the restricted nullcone

$$\mathcal{N}_1(\mathfrak{g}) = \{x \in \mathfrak{g} : x^{[p]} = 0\}.$$

Moreover, Friedlander and Parshall showed that

$$\mathcal{V}_{G_1}(M) \cong \{x \in \mathfrak{g} : x^{[p]} = 0, M|_{u(\langle x \rangle)} \text{ is not free}\} \cup \{0\}.$$

If G is a reductive algebraic group over k and $\mathfrak{g} = \text{Lie } G$ then $\mathcal{N}_1(\mathfrak{g})$ is a G -stable conical subvariety inside the cone of nilpotent elements (nullcone) $\mathcal{N} := \mathcal{N}(\mathfrak{g})$. In the special case when $G = GL_n(k)$,

$$\mathcal{N}_1(\mathfrak{g}) = \{A \in \mathfrak{gl}_n(k) : A^p = 0\}.$$

The group $GL_n(k)$ acts on \mathcal{N} via conjugation and has finitely many orbits (indexed by partitions of n). Note that $\mathcal{N}_1(\mathfrak{g})$ is G -stable. For general reductive groups G , the nullcone \mathcal{N} has been well studied (see [Car] [CM] [Hu2]) because of its beautiful geometric properties with deep connections to representation theory. The group G acts on \mathcal{N} via the adjoint representation and \mathcal{N} has finitely many G -orbit (which are classified). In this setting support varieties behave well with respect to the G -action.

$$(3.3.6) \quad \text{If } M \text{ is a } G\text{-module, then } \mathcal{V}_A(M) \text{ is } G\text{-invariant.}$$

(3.3.7) If $M \in \text{mod}(G)$, $\mathcal{V}_A(M) = \overline{G \cdot x_1} \cup \overline{G \cdot x_2} \cup \cdots \cup \overline{G \cdot x_s}$. There are only finitely many possibilities for $\mathcal{V}_A(M)$ in this case because there are finitely many nilpotent G -orbits.

4. RELATING SUPPORT VARIETIES

4.1. Motivation: When I was a graduate student at Yale, my Ph.D advisor George Seligman told me a story about the Ph.D thesis of his former student David Pollack (circa 1966). Seligman said that he proposed the following problem to Pollack for his thesis: to determine the representation type for the restricted enveloping algebras $u(\mathfrak{g})$ where $\mathfrak{g} = \text{Lie } G$ and G is a reductive algebraic group. More specifically, at the time the question was to determine whether these algebras have finite or infinite representation type.

For group algebras this problem was settled earlier: a group algebra over a field of characteristic $p > 0$ has finite representation type if and only if its p -Sylow subgroups are cyclic. Therefore, when $G = SL_2(k)$, and $A = kG(\mathbb{F}_p)$, all Sylow subgroups are cyclic and conjugate to the subgroup of unipotent upper triangular matrices. Consequently, $A = kSL_2(\mathbb{F}_p)$ has finite representation type.

One of the first challenges was for Pollack to determine the representation type of $u(\mathfrak{sl}_2)$. At the time Walter Feit thought by analogy that this algebra should be of finite representation type. But, after some efforts in trying to prove this, Pollack constructed the projective indecomposables for the algebra, and using these concrete realizations he constructed infinitely many indecomposable modules for $u(\mathfrak{sl}_2)$. Moreover, Pollack was able to show that $u(\mathfrak{g})$ where $\mathfrak{g} = \text{Lie } G$ (G is a reductive algebraic group) has infinite representation type.

After hearing this story, I thought that there should be direct connections between understanding representation type between finite Chevalley groups and their associated Lie algebras. The pursuit in finding such connections was in part a motivating factor in the development of the results outlined in this section.

4.2. Parshall Conjecture: In 1987, Parshall (another former student of Seligman) asked the following question in his article in the Arcata Conference Proceedings. Let G be a reductive algebraic group and M be in $\text{mod}(G)$. If M is projective as a G_1 -module then is M projective over $G(\mathbb{F}_p)$? The affirmative version to this question has become known as the ‘‘Parshall Conjecture’’. As we have seen in Section 2.3 the Parshall Conjecture holds if $p \geq 2(h - 1)$.

For this section, we will need some additional notion. Let B be a Borel subgroup of G , U be the unipotent radical of B . Furthermore, let $G(\mathbb{F}_p)$, $B(\mathbb{F}_p)$, and $U(\mathbb{F}_p)$ be the corresponding finite groups, and $\mathfrak{g} = \text{Lie } G$, $\mathfrak{b} = \text{Lie } B$, and $\mathfrak{n} = \text{Lie } U$ be the corresponding Lie algebras.

We will now describe a process which allows us to pass from $\text{mod}(G_1)$ to $\text{mod}(G(\mathbb{F}_p))$ without lifting to $\text{mod}(G)$. First consider

$$U(\mathbb{F}_p) = \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_n = \{1\}$$

the lower central series where $\Gamma_i = [\Gamma_{i-1}, U(\mathbb{F}_p)]$. Then the associated graded object $\text{gr } U(\mathbb{F}_p) = \bigoplus_{n \geq 1} \Gamma_n / \Gamma_{n+1}$ is a restricted Lie algebra (for most primes). We can identify the associated graded Lie algebra with \mathfrak{n} . That is, $\text{gr } U(\mathbb{F}_p) \cong \mathfrak{n}$.

Now consider filtering the group algebra of $U(\mathbb{F}_p)$ with powers of its augmentation ideal:

$$kU(\mathbb{F}_p) = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_t = \{0\}$$

where $I_1 = \text{rad } kU(\mathbb{F}_p)$. One can now form the associated graded algebra $\text{gr } kU(\mathbb{F}_p) = \bigoplus_{n \geq 0} I_n/I_{n+1}$.

By using results of Quillen and Jennings, we have the following isomorphism of algebras:

$$\text{gr } kU(\mathbb{F}_p) \cong u(\text{gr } U(\mathbb{F}_p)) \otimes_{\mathbb{F}_p} k \cong u(\mathfrak{n}).$$

The filtration on the group algebra of $U(\mathbb{F}_p)$ can be used to construct the May spectral sequence. For M in $\text{mod}(U(\mathbb{F}_p))$,

$$E_1^{i,j} = H^{i+j}(u(\mathfrak{n}), \text{gr } M) \Rightarrow H^{i+j}(U(\mathbb{F}_p), M).$$

Here we are using the identification: $\text{gr } kU(\mathbb{F}_p) = u(\mathfrak{n})$. If M in $\text{mod}(B)$, then $\text{gr } M = M|_{U_1}$ and we can rewrite the spectral sequence as

$$E_1^{i,j} = H^{i+j}(U_1, M) \Rightarrow H^{i+j}(U(\mathbb{F}_p), M).$$

This shows that if M in $\text{mod}(B)$ then $c_{U_1}(M) \geq c_{U(\mathbb{F}_p)}(M)$. Moreover, one obtains the following theorem.

Theorem 4.2.1. [LN1, Thm. 3.4] *Let M be in $\text{mod}(G)$. Then $c_{G(\mathbb{F}_p)}(M) \leq \frac{1}{2}c_{G_1}(M)$.*

Proof. (Sketch) First $c_{G(\mathbb{F}_p)}(M) = c_{U(\mathbb{F}_p)}(M)$ because $U(\mathbb{F}_p)$ is a Sylow subgroup of $G(\mathbb{F}_p)$. Next $c_{U_1}(M) = c_{B_1}(M) = \frac{1}{2}c_{G_1}(M)$ by using a result of Spaltenstein which says that the intersection of a G -orbit inside the nilpotent cone with \mathfrak{n} is a union of B -stable sets having one-half the dimension of the original orbit. Finally, one can apply the inequality $c_{U_1}(M) \geq c_{U(\mathbb{F}_p)}(M)$. \square

As a corollary, one can obtain a proof of the Parshall Conjecture.

Corollary 4.2.2. *Let M be in $\text{mod}(G)$. If M is a projective G_1 -module then M is a projective $kG(\mathbb{F}_p)$ -module.*

Proof. If M is projective as a G_1 -module then $c_{G_1}(M) = 0$. Therefore, $c_{G(\mathbb{F}_p)}(M) = 0$ by using the complexity bound, thus M is a projective $kG(\mathbb{F}_p)$ -module. \square

Let us consider these results in the context of the representation type question in Section 4.1. Since $kSL_2(\mathbb{F}_p)$ has finite representation type we have $c_{SL_2(\mathbb{F}_p)}(k) = 1$. Therefore, $c_{(SL_2)_1}(k) \geq 2$ which demonstrates immediately that $(SL_2)_1$ has infinite representation type because the syzygies $\Omega^n(k)$ are indecomposable modules and must grow in dimension at least linearly in n .

4.3. Partial Converse to the Parshall Conjecture: A natural question to ask is whether the converse to the Parshall Conjecture holds, that is if M is in $\text{mod}(G)$, and M is projective over $kG(\mathbb{F}_p)$, is it projective over G_1 ?

The immediate answer is no. Let $M = St^{(1)} = L((p-1)\rho)^{(1)}$. Then M is projective as $G(\mathbb{F}_p)$ -module, and $M \simeq \oplus k$ as G_1 -module. Let \mathcal{C}_p be the full subcategory of G -modules whose composition factors have highest weight λ , satisfying $\langle \lambda, \tilde{\alpha}^\vee \rangle \leq p(p-1)$ where $\tilde{\alpha}$ is the highest root. The converse is valid as long as we restrict the collection of G -modules under consideration.

Theorem 4.3.1. [LN1, Thm. 4.4] *Let $M \in \mathcal{C}_p$. If M is projective over $kG(\mathbb{F}_p)$ then M is projective over G_1 .*

Friedlander and Parshall proved that can one check projectivity over G_1 for modules in $\text{mod}(G)$ by considering the restriction to the one-dimensional Lie algebra spanned by a long root vector.

Theorem 4.3.2. [FP1, (2.4b) Prop.] *Let $\alpha \in \Phi$ be long root and M be in $\text{mod}(G)$. Then M is projective over G_1 if and only if $M|_{u(\langle x_\alpha \rangle)}$ is projective.*

We can now apply Friedlander and Parshall's result above along with the validity of the Parshall Conjecture to prove a projectivity criterion over $G(\mathbb{F}_p)$ for modules in \mathcal{C}_p .

Corollary 4.3.3. [LN1, Thm 4.5] *Let M be in \mathcal{C}_p . The following are equivalent:*

- (a) M is projective over $kG(\mathbb{F}_p)$;
- (b) $M|_{x_\alpha(\mathbb{F}_p)}$ is projective for all $\alpha \in \Phi$;
- (c) $M|_{x_\beta(\mathbb{F}_p)}$ is projective for $\beta \in \Phi$ long.

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are clear. We need to show that $(c) \Rightarrow (a)$. Suppose that $M|_{x_\beta(\mathbb{F}_p)}$ is projective. Then $M|_{SL_2(\mathbb{F}_p)_\beta}$ is projective where $SL_2(\mathbb{F}_p)_\beta$ is the semisimple part of the subgroup generated by $x_\beta(\mathbb{F}_p)$ and $x_{-\beta}(\mathbb{F}_p)$. By the partial converse to the Parshall Conjecture, $M|_{u(\mathfrak{sl}_2)_\beta}$ is projective. Therefore, $M|_{u(\langle x_\beta \rangle)}$ is projective. Friedlander and Parshall's result implies that M is projective as G_1 -module. The Parshall Conjecture can be employed to conclude that M is projective as $G(\mathbb{F}_p)$ -module. \square

4.4. A Map Between Support Varieties: Parshall also posed the problem in his Arcata Conference Proceedings article to find a relationship between the support varieties for G_1 and $G(\mathbb{F}_p)$. The results in this section are due to the Carlson, Lin and the author [CLN].

Let $\mathcal{V}_{G_1}(M)$ denote the support variety of M in $\text{mod}(G_1)$. Recall that

$$\mathcal{V}_{G_1}(M) = \{x \in \mathfrak{g} : x^{[p]} = 0, M|_{u(\langle x \rangle)} \text{ is not free}\} \cup \{0\} \subseteq \mathcal{N}_1(\mathfrak{g}) \subseteq \mathcal{N}.$$

We will assume that p is good (i.e., if $\beta \in \Phi^+$ with $\beta = \sum_{\alpha \in \Delta} n_\alpha \alpha$ then $p \nmid n_\alpha$ for any α). For example, $\Phi = E_6$, p is good except when $p = 2, 3$. The following theorem provides a description of $\mathcal{V}_{G_1}(k) \cong \mathcal{N}_1(\mathfrak{g})$ when p is good.

Theorem 4.4.1. [NPV, (6.2.1) Thm.] *Let p be good. Then there exist $J \subseteq \Delta$ such that $\mathcal{N}_1(\mathfrak{g}) = G \cdot \mathfrak{u}_J$, where $\mathfrak{g} = \mathfrak{u}_J \oplus \mathfrak{l}_J \oplus \mathfrak{u}_J^+$ where \mathfrak{l}_J is the corresponding Levi subalgebra. Furthermore, the nilpotence degree of $\mathfrak{u}_J < p$.*

Let \mathcal{U} be the unipotent variety, which is the set of unipotent element in G , and set

$$\mathcal{U}_1 = \{u \in \mathcal{U} : u^p = 1\}.$$

When p is good, $\mathcal{U}_1 = G \cdot U_J$ where U_J is the unipotent radical of the parabolic subgroup associated to J (as in the preceding theorem). By using variations on the exponential map, Seitz was able to prove the following theorem.

Theorem 4.4.2. [Sei, Prop. 5.1] *There exists a P_J -equivariant isomorphism of algebraic k -varieties:*

$$\exp : \mathfrak{u}_J \rightarrow \mathcal{U}_J$$

which is defined over $\mathbb{Z}_{(p)}$.

Under suitable normality conditions, we were able to extend Seitz's map to a G -equivariant isomorphism of varieties.

Theorem 4.4.3. [CLN, Thm. 3] *Let $\mathcal{N}_1(g)$ be normal. Then \exp extends to a G -equivariant isomorphism:*

$$\exp : G \cdot \mathfrak{u}_J \rightarrow G \cdot U_J.$$

From now on let us define

$$\log : \mathcal{U}_1 \rightarrow \mathcal{N}_1(\mathfrak{g})$$

as the inverse of \exp . For M a $G(\mathbb{F}_p)$ -module, let $\mathcal{V}_{G(\mathbb{F}_p)}(M)$ denote the support variety of M . One can describe the support variety of the trivial module as

$$\mathcal{V}_{G(\mathbb{F}_p)}(k) = \lim_{\overline{E}} \mathcal{V}_E(k),$$

where E ranges over the elementary abelian p -subgroups. Now one can identify $\mathcal{V}_E(k)$ with a rank variety $\mathcal{V}_E^{\text{rank}}(k)$ which allows one to construct a map $\mathcal{V}_E(k) \hookrightarrow \mathcal{U}_1$. By putting all these maps together we have

$$\mathcal{V}_{G(\mathbb{F}_p)}(k) \hookrightarrow \mathcal{U}_1/G(\mathbb{F}_p) \rightarrow \mathcal{N}_1(\mathfrak{g})/G(\mathbb{F}_p) \cong V_{G_1}(k)/G(\mathbb{F}_p).$$

Therefore, we have defined a map of support varieties:

$$\psi : \mathcal{V}_{G(\mathbb{F}_p)}(k) \rightarrow \mathcal{V}_{G_1}(k)/G(\mathbb{F}_p).$$

Now define the category \mathcal{D} as follows. Let $l = \text{rank}(\Phi)$ and $(b_{ij}) = (\langle \alpha_i, \alpha_j^\vee \rangle)^{-1}$. Let \mathcal{D} be the full subcategory of G -module whose composition factors have high weight satisfying

$$\sum_{i=1}^l \sum_{j=1}^l \langle \lambda, \alpha_i^\vee \rangle b_{ij} < \frac{p(p-1)}{2}.$$

The following theorem gives an explicit description of the image of ψ on $\mathcal{V}_{G(\mathbb{F}_p)}(M)$.

Theorem 4.4.4. [CLN, Cor. 1.2] *Let G be a simple algebra group scheme defined over \mathbb{F}_p . Moreover, let $\mathcal{N}_1(g)$ be normal and $M \in \mathcal{D}$, then*

$$\psi(\mathcal{V}_{G(\mathbb{F}_p)}(M)) \subseteq \mathcal{V}_{G_1}(M)/G(\mathbb{F}_p).$$

More precisely:

$$\psi(\mathcal{V}_{G(\mathbb{F}_p)}(M)) = \{x \in \mathcal{N}_1^{\mathbb{F}_p} : x^{[p]} = 0, M|_{u(\langle x \rangle)} \text{ is not free}\} \cup \{0\}$$

where $\mathcal{N}_1^{\mathbb{F}_p}$ is the set of \mathbb{F}_p -expressible elements (cf. [CLN, Section 3]).

4.5. Applications: The results in the preceding section can be used to understand projectivity for finite groups of Lie type.

Theorem 4.5.1. *Let G be a simple algebraic group, M be in \mathcal{C}_p which is not projective over $kG(\mathbb{F}_p)$. Then*

- (a) *If M is indecomposable in $\text{mod}(kG)$ and the rank of Φ is greater than or equal to 2 then $\dim \mathcal{V}_{G(\mathbb{F}_p)}(M) \geq 2$;*
- (b) *If $\Phi = A_n$ then $\dim \mathcal{V}_{GL_n(\mathbb{F}_p)}(M) \geq n$.*

We can also determine the dimension of support varieties for simple modules for rank 2 groups.

Theorem 4.5.2. *Let G be a simple algebraic group where Φ has rank 2 (i.e., $\Phi = A_2$ ($p > 2$), B_2 ($p > 7$), G_2 ($p > 19$), and let $\lambda \in X_1(T)$.*

- (a) *If $\lambda = (p-1)\rho$ then $\mathcal{V}_{G(\mathbb{F}_p)}(L(\lambda)) = \{0\}$.*
- (b) *If $\lambda \neq (p-1)\rho$ then*

$$\dim \mathcal{V}_{G(\mathbb{F}_p)}(L(\lambda)) = \begin{cases} 2 & \Phi = A_2 \\ 3 & \Phi = B_2, G_2. \end{cases}$$

4.6. Generalizations: Friedlander [F2] has recently found an appropriate generalization of many of the aforementioned results for $G(\mathbb{F}_q)$ where $r \geq 1$. The idea involves using base changes on the Lie algebra level and to view $\mathfrak{g}_{\mathbb{F}_q}$ as an \mathbb{F}_p -Lie algebra. Set $A_q = u(\mathfrak{g}_{\mathbb{F}_q} \otimes_{\mathbb{F}_p} k)$. In particular, Friedlander proves a generalization of the Parshall Conjecture.

Theorem 4.6.1. [F2, Cor. 4.4] *Let $M \in \text{Mod}(G)$. If $M \downarrow_{A_q}$ is projective then $M|_{G(\mathbb{F}_q)}$ is projective.*

Furthermore, by using the Π -point theory developed by Friedlander and Pevtsova [FPe], he constructs a map

$$\psi : \mathcal{V}_{G(\mathbb{F}_q)}(k) \hookrightarrow \mathcal{V}_{A_q}(k)/G(\mathbb{F}_q)$$

(cf. [F2, Corollary 3.6]).

5. RELATING COHOMOLOGY

5.1. Spectral Sequences via Truncation: We will next investigate questions which involve relating extensions in the three categories $\text{Mod}(G(\mathbb{F}_q))$, $\text{Mod}(G)$, and $\text{Mod}(G_r)$. The results were obtained in [BNP1, BNP2, BNP3] with generalizations to the twisted cases in [BNP5, BNP6].

The basic idea is to first construct spectral sequences which connect the cohomology theories of $\text{Mod}(G(\mathbb{F}_q))$ and $\text{Mod}(G)$. The basic problem is that $\text{Mod}(G)$ does not have enough projective objects. Therefore, it is better to work with “truncated categories” obtained via saturated sets of weights. These truncated categories are highest weight categories and Morita equivalent to the module category for some quasi-hereditary algebra (cf. [CPS]). The cohomology for these quasi-hereditary algebras provides a better approximation of the homological behavior in $\text{Mod}(G(\mathbb{F}_q))$. Once we provide a connection to the G -category, we apply the Lyndon-Hochschild-Serre (LHS) spectral sequence and use information about the cohomology in $\text{Mod}(G_r)$, in addition to the “geometry of the flag variety G/B ” to obtain results about the cohomology for the finite Chevalley groups.

Let $\Pi = \{\lambda \in X(T)_+ : \langle \lambda + \rho, \alpha_0^\vee \rangle \leq 2p^r \langle \rho, \alpha_0^\vee \rangle\}$ and \mathcal{C} be the full subcategory of modules in $\text{Mod}(G)$ whose composition factors have highest weight in Π . Let $\mathcal{T} : \text{Mod}(G) \rightarrow \mathcal{C}$ be the functor defined where $\mathcal{T}(M)$ is the largest submodule of M contained in \mathcal{C} . The functor \mathcal{T} is left exact.

Now define the functor $\mathcal{G} : \text{Mod}(G(\mathbb{F}_q)) \rightarrow \mathcal{C}$ as

$$\mathcal{G}(N) = \mathcal{T}(\text{ind}_{G(\mathbb{F}_q)}^G N)$$

where $N \in \text{Mod}(G(\mathbb{F}_q))$. Since $G/G(\mathbb{F}_q)$ is affine, the induction functor is exact. Therefore, \mathcal{G} is a left exact functor and admits higher right derived functors $R^j \mathcal{G}(-)$ for $j \geq 0$.

The following theorem provides a spectral sequence which relates extensions in $\text{Mod}(G(\mathbb{F}_q))$ and \mathcal{C} .

Theorem 5.1.1. *Let M in \mathcal{C} and N in $\text{Mod}(G(\mathbb{F}_q))$. Then there exists a first quadrant spectral sequence:*

$$E_2^{i,j} = \text{Ext}_{\mathcal{C}}^i(M, R^j \mathcal{G}(N)) \Rightarrow \text{Ext}_{G(\mathbb{F}_q)}^{i+j}(M, N).$$

We note that since \mathcal{C} is obtained from a saturated set of weight, we have

$$\text{Ext}_{\mathcal{C}}^i(M_1, M_2) = \text{Ext}_G^i(M_1, M_2)$$

for all M_1, M_2 in \mathcal{C} and $i \geq 0$. One can apply this spectral sequence along with the lifting property of projective G_r -modules to obtain the following corollary.

Corollary 5.1.2. *Let $p \geq 2(h-1)$ with $\lambda, \mu \in X_r(T)$*

- (a) $\text{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\mu)) \cong \text{Ext}_G^1(L(\lambda), \mathcal{G}(L(\mu)))$;
- (b) $\text{Ext}_G^2(L(\lambda), \mathcal{G}(L(\mu))) \hookrightarrow \text{Ext}_{G(\mathbb{F}_q)}^2(L(\lambda), L(\mu))$.

Proof. Let $M = L(\lambda)$, $N = L(\mu)$ in the spectral sequence. The spectral sequence yields a five term exact sequence:

$$0 \rightarrow E_2^{1,0} \rightarrow E_1 \rightarrow E_2^{0,1} \rightarrow E_2^{2,0} \rightarrow E_2.$$

In order to prove the result we need to show that $E_2^{0,1} = 0$. It suffices to prove that $R^1 \mathcal{G}(L(\mu))$ has no composition factors isomorphic to $L(\lambda)$. When $p \geq 2(h-1)$, the projective cover $P(\lambda)$ of $L(\lambda)$ in \mathcal{C} is isomorphic to $Q_r(\lambda)$. This fact uses the lifting property of projective G_r -modules. Moreover, one can apply the spectral sequence with $M = P(\lambda)$ and $N = L(\mu)$. The spectral sequence then collapses and yields:

$$\text{Hom}_{\mathcal{C}}(P(\lambda), R^1 \mathcal{G}(L(\mu))) \cong \text{Ext}_{G(\mathbb{F}_q)}^1(P(\lambda), L(\mu)).$$

By putting all of this information together, we have

$$\begin{aligned} [R^1 \mathcal{G}(L(\mu)) : L(\lambda)] &= \dim \text{Hom}_{\mathcal{C}}(P(\lambda), R^1 \mathcal{G}(L(\mu))) \\ &= \dim \text{Ext}_{G(\mathbb{F}_q)}^1(P(\lambda), L(\mu)) \\ &= \dim \text{Ext}_{G(\mathbb{F}_q)}^1(Q_r(\lambda), L(\mu)) \\ &= 0. \end{aligned}$$

□

5.2. Extensions between simple modules: Our knowledge about extensions involving two simple modules in $\text{Mod}(G)$, $\text{Mod}(G_r)$ and $\text{Mod}(G(\mathbb{F}_q))$ is very minimal. We can state this in terms of three open problems:

(5.2.1) Determine $\text{Ext}_G^j(L(\sigma_1), L(\sigma_2))$ for $\sigma_1, \sigma_2 \in X(T)_+$, $j \geq 0$.

(5.2.2) Determine $\text{Ext}_{G_r}^j(L(\sigma_1), L(\sigma_2))$ for $\sigma_1, \sigma_2 \in X_r(T)$, $j \geq 0$.

(5.2.3) Determine $\text{Ext}_{G(\mathbb{F}_q)}^j(L(\sigma_1), L(\sigma_2))$ for $\sigma_1, \sigma_2 \in X_r(T)$, $j \geq 0$.

The Lusztig Conjecture predicts the characters of the simple G -modules when $p \geq h$. If the Lusztig Conjecture holds then (5.2.1) and (5.2.2) can be solved for regular weights. Moreover, if there are certain vanishing/non-vanishing conditions that hold when $j = 1$ for the extension groups in (5.2.1) and (5.2.2) then the Lusztig Conjecture is valid.

It is interesting to consider the case when $L(\sigma_1) \cong L(\sigma_2) \cong k$. Then $H^j(G, k) = 0$ for $j > 0$ and is isomorphic to k when $j = 0$. For the Frobenius kernels, $H^{2\bullet}(G_1, k) = k[\mathcal{N}]$ where \mathcal{N} is the nilpotent cone, and $H^{2\bullet+1}(G_1, k) = 0$ when $p > h$ (cf. [AJ, FP2]). The cohomology for G_r is not known in general for $r > 1$. Finally, very little is known about the cohomology groups $H^\bullet(G(\mathbb{F}_q), k)$. We will investigate questions about these groups later in Section 6.

In the case when $\sigma = \sigma_1 = \sigma_2$ and $j = 1$ one is looking at the case of self-extensions. By standard arguments [Jan1, II 2.12], one can show that $\text{Ext}_G^1(L(\sigma), L(\sigma)) = 0$ for $\sigma \in X(T)_+$. In 1984, Andersen [And1, Theorem 4.5] proved the following result about self extensions for G_r :

Theorem 5.2.1. *Suppose $\Phi \neq C_n$, when $p = 2$. Then $\text{Ext}_{G_r}^1(L(\sigma), L(\sigma)) = 0$ for all $\sigma \in X_r(T)$.*

Andersen used information about the structure of the higher line bundle cohomology groups in his calculation. The use of these geometric techniques will be a dominant theme in our examination of self-extensions and cohomology for finite Chevalley groups

5.3. Ext^1 -formulas for simple $G(\mathbb{F}_q)$ -modules: We will now run through a series of steps which will lead us to produce Ext^1 -formulas for finite Chevalley groups.

(1) Let $\lambda, \mu \in X_r(T)$, then

$$\text{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\mu)) \cong \text{Ext}_G^1(L(\lambda), L(\mu) \otimes \mathcal{G}(k))$$

for all $\lambda, \mu \in X_r(T)$ [BNP3, Theorem 2.2].

(2) We can explicitly describe $\mathcal{G}(k)$ as a G -module. Set $\Gamma_{2h-1} = \{\nu \in X(T)_+ : \langle \nu, \alpha_0^\vee \rangle < 2h-1\}$. Then a miracle happens! For $p \geq 3(h-1)$, $\mathcal{G}(k)$ is semisimple [BNP1, Theorem 7.4]. Moreover,

$$\begin{aligned} \mathcal{G}(k) &\cong \bigoplus_{\nu \in \Gamma_{2h-1}} L(\nu) \otimes [L(\nu)^{(r)}]^* \\ &\cong \bigoplus_{\nu \in \Gamma_{2h-1}} L(\nu) \otimes L(\nu^*)^{(r)}. \end{aligned}$$

(3) Set $\Gamma = \{\nu \in X(T)_+ : \langle \nu, \alpha_0^\vee \rangle < h-1\}$. By using the decomposition of $\mathcal{G}(k)$ along with some additional information about G -extensions, we obtain the following formula.

Theorem 5.3.1. *Let $p \geq 3(h-1)$ and $\lambda, \mu \in X_r(T)$. Then*

$$\text{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\mu)) \cong \bigoplus_{\nu \in \Gamma} \text{Ext}_G^1(L(\lambda) \otimes L(\nu)^{(r)}, L(\mu) \otimes L(\nu)).$$

We can now refine the aforementioned Ext^1 -formula when $r \geq 2$ and $r = 1$. First let $r \geq 2$. For $\sigma \in X_r(T)$ with $\sigma = \sigma_0 + \sigma_1 p + \cdots + \sigma_{r-2} p^{r-2} + \sigma_{r-1} p^{r-1}$ set $\widehat{\sigma} = \sigma_0 + \sigma_1 p + \cdots + \sigma_{r-2} p^{r-2}$. We can apply the LHS spectral sequence twice with $G_{r-1} \trianglelefteq G$, and $G_1 \trianglelefteq G$ to obtain the following theorem.

Theorem 5.3.2. *Let $p \geq 3(h-1)$, $\lambda, \mu \in X_r(T)$, $r \geq 2$. Then*

$$\text{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\mu)) = \text{Ext}_G^1(L(\lambda), L(\mu)) \oplus R$$

where $R = \bigoplus_{\nu \in \Gamma - \{0\}} \text{Hom}_G(L(\nu), \text{Ext}_{G_1}^1(L(\lambda_{r-1}), L(\mu_{r-1}))) \otimes \text{Hom}_G(L(\widehat{\lambda}), L(\widehat{\mu}) \otimes L(\nu)).$

One can also produce a formula when $r = 1$. In these formulas one should observe how the information from the Frobenius kernels is playing a role in the determination of cohomology for the finite Chevalley groups.

Theorem 5.3.3. *Let $p \geq 3(h - 1)$ and $\lambda, \mu \in X_1(T)$. Then*

$$\mathrm{Ext}_{G(\mathbb{F}_p)}^1(L(\lambda), L(\mu)) \cong \mathrm{Ext}_G^1(L(\lambda), L(\mu)) \oplus R$$

where

$$R = \bigoplus_{\nu \in \Gamma - \{0\}} \mathrm{Hom}_G(L(\nu), \mathrm{Ext}_{G_1}^1(L(\lambda), L(\mu) \otimes L(\nu))^{(-1)}).$$

5.4. Self extensions: For $r \geq 2$, the groups $G(\mathbb{F}_q)$ do not admit self extensions when $p \geq 3(h - 1)$.

Theorem 5.4.1. *Let $p \geq 3(h - 1)$, $r \geq 2$ and $\lambda \in X_r(T)$. Then $\mathrm{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\lambda)) = 0$.*

Proof. As we have seen $\mathrm{Ext}_G^1(L(\lambda), L(\lambda)) = 0$. Therefore, we need to prove that $R = 0$ in Theorem 5.3.2. But, this follows by applying Andersen's result on self-extensions for G_1 . \square

For $r = 1$, Humphreys proved the existence of self extensions when the root system is of type C_n for general p . One can use a more detailed analysis using the cohomology of line bundles to obtain the following result (cf. [BNP3, Section 4]).

Theorem 5.4.2. *Let $p \geq 3(h - 1)$ and $\lambda \in X_1(T)$. If either*

- (a) *G does not have underlying root system of type A_1 or C_n or*
- (b) *$\langle \lambda, \alpha_n^\vee \rangle \neq \frac{p-2-c}{2}$, where α_n is the unique long simple root and c is odd with $0 < |c| \leq h - 1$,*

then $\mathrm{Ext}_{G(\mathbb{F}_p)}^1(L(\lambda), L(\lambda)) = 0$.

5.5. Applications: Our results involving Ext^1 for finite Chevalley group enabled us to address several open questions.

(1) Smith asked the following question (Question 5) in his 1985 paper [Sm]:

Question 5: Let V be a $G(\mathbb{F}_q)$ -module satisfying Hypothesis A with all composition factors isomorphic. Is V completely reducible?

Our self extension results stated in Section 5.4 answer Smith's question independent of "Hypothesis A".

(2) Cline, Parshall, Scott, and vanderKallen's [CPSvdK] famous result on rational and generic cohomology can be stated as follows. Let V in $\mathrm{Mod}(G)$. For fixed $i \geq 0$, consider the restriction map in cohomology:

$$H^i(G, V^{(s)}) \xrightarrow{\mathrm{res}} H^i(G(\mathbb{F}_q), V^{(s)}).$$

When r and s are sufficiently large, the map res is an isomorphism.

We can see the phenomenon in our Ext^1 -formulas. Let $p \geq 3(h - 1)$. Fix $\lambda, \mu \in X_r(T)$. As r get large $\lambda_{r-1} = \mu_{r-1} = 0$, and $\mathrm{Ext}_{G_1}^1(k, k) = 0$. Therefore, by Theorem 5.3.2, $R = 0$ and

$$\mathrm{Ext}_G^1(L(\lambda), L(\mu)) \xrightarrow{\mathrm{res}} \mathrm{Ext}_{G(\mathbb{F}_q)}^1(L(\lambda), L(\mu))$$

is an isomorphism. Observe that in this instance no Frobenius twists are necessary.

(3) Jantzen [Jan4] proved the following result which insured that the restriction map in cohomology is an isomorphism given upper bounds on the highest weight of the composition factors of V .

Theorem 5.5.1. *Let $V \in \text{Mod}(G)$ where the composition factors $L(\mu)$ satisfy*

$$\langle \mu, \alpha_0^\vee \rangle \leq \begin{cases} p^r - 3p^{r-1} - 3 & \Phi = G_2 \\ p^r - 2p^{r-1} - 2 & \Phi \neq G_2. \end{cases}$$

Then res is an isomorphism.

For large primes, Andersen [And2] gives uniform conditions on the high weight on the composition factors to insure that the restriction map is an isomorphism.

Theorem 5.5.2. *Let $p \geq 3(h-1)$ and suppose that the composition factors of V satisfy*

$$\langle \mu, \alpha_0^\vee \rangle \leq p^r - p^{r-1} - 2 \quad (\Phi \neq A_1).$$

Then res is an isomorphism.

Our results [BNP2, Theorem 4.8(A)] enabled us to give the best possible bounds to insure that the restriction map is an isomorphism when $r \geq 2$. The optimal bounds when $r = 1$ are still unknown.

Theorem 5.5.3. *Let G be a simple, simply connected algebraic group with $p \geq 3(h-1)$, $r \geq 2$. If V has composition factors $L(\mu)$ satisfying*

$$\langle \mu, \alpha_0^\vee \rangle \leq \begin{cases} p^r - 2p^{r-1} & \Phi = A_1 \\ p^r - p^{r-1} & \Phi = A_n \\ p^r & \Phi = B_n, C_n, D_n, E_6, E_7 \\ 2p^r - p^{r-1} + 1 & \Phi = E_8, F_4, G_2. \end{cases}$$

Then res is an isomorphism.

6. COMPUTING COHOMOLOGY FOR FINITE GROUPS OF LIE TYPE

6.1. Vanishing Ranges: In 2005 there was a Conference on the Cohomology of Finite Groups at Oberwolfach. At the beginning of Eric Friedlander's talk he started by stating that we know very little about the cohomology $H^\bullet(G(\mathbb{F}_q), k)$ where $\text{char } k = p > 0$ other than Quillen's description of the maximal ideal spectrum. Friedlander went on to say that it would be nice to know when the cohomology starts (i.e., in which degree the first non-trivial cohomology lives). There are two main aspects to this problem.

(6.1.1) Vanishing Ranges: Locating $D > 0$ such that the cohomology groups $H^i(G(\mathbb{F}_q), k) = 0$ for $0 < i < D$.

(6.1.2) Determining the First Non-Trivial Cohomology Class: Find a D such that $H^i(G(\mathbb{F}_q), k) = 0$ for $0 < i < D$ and $H^D(G(\mathbb{F}_q), k) \neq 0$. A D satisfying this property will be called a *sharp bound*.

Quillen computed the cohomology of $G(\mathbb{F}_q)$ in the non-describing characteristic and provided a vanishing range (6.1.1) in the case of $GL_n(\mathbb{F}_q)$ in the describing characteristic case (cf. [Q3]). Later Friedlander [F1] and Hiller [H] discovered vanishing ranges for other groups of Lie type.

The aim of this section is to present some recent work of the author with Bendel and Pillen. We solve (6.1.2) for groups of types A_n and C_n where $p > h$ and $r = 1$. In order to obtain these results we first employ a variation on the ideas presented in Section 5 by investigating the induction $\mathcal{G}_r(k) = \text{ind}_{G(\mathbb{F}_q)}^G k$. We indicate that $\mathcal{G}_r(k)$ has a natural filtration as a $G \times G$ -module. This allows us to reduce our problem to looking at sections of the filtration (i.e., modules of the form $H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}$, $\lambda \in X(T)_+$). The next step is to apply the LHS spectral sequence in the case when $r = 1$. The spectral sequence collapses in the case when $p > h$ by using results due to Kumar, Lauritzen and Thompsen [KLT]. This in turn provides a method for giving an upper bound for the cohomology $H^\bullet(G(\mathbb{F}_p), k)$ using the combinatorics of the nilpotent cone via Kostant's Partition Functions. The steps are presented in the following diagram. The results and details in this section can be found in [BNP8, BNP9].

$$\begin{array}{ccccc}
\text{Induction} & & & & \\
\text{Functor} & & & & \\
H^i(G(\mathbb{F}_q), k) & \rightarrow & H^i(G, \mathcal{G}_r(k)) & & \\
& \downarrow & \text{Filtrations} & & \\
H^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) & \rightarrow & H^i(G_1, H^0(\lambda)) & \rightarrow & \text{Root Combinatorics.} \\
& \text{LHS Spectral} & \text{Kostant Partition} & & \\
& \text{Sequences} & \text{Functions} & &
\end{array}$$

6.2. Induction and Filtrations: Let $\mathcal{G}_r = \text{ind}_{G(\mathbb{F}_q)}^G(-)$. This functor is exact and all its higher right derived functors vanish. Therefore, we can apply Frobenius reciprocity to obtain the following isomorphism of extension groups.

Proposition 6.2.1. *Let M, N be in $\text{Mod}(G)$. Then, for all $i \geq 0$,*

$$\text{Ext}_{G(\mathbb{F}_q)}^i(M, N) \cong \text{Ext}_G^i(M, N \otimes \mathcal{G}_r(k)).$$

As a $G \times G$ -module the coordinate algebra $k[G]$ has a filtration with sections of the form $H^0(\lambda) \otimes H^0(\lambda^*)$ where $\lambda \in X(T)_+$. Each section of this form appears exactly one time. By using the Lang map, we were able to show that $\mathcal{G}_r(k) = k[G/G(\mathbb{F}_q)]$ admits a natural filtration as a $G \times G$ -module.

Proposition 6.2.2. *The $G \times G$ -module $\mathcal{G}_r(k)$ has a filtration with factors of the form $H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}$ with multiplicity one for each $\lambda \in X(T)_+$.*

6.3. The existence of a filtration on $\mathcal{G}_r(k)$ with sections of the form $H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}$, $\lambda \in X(T)_+$, in addition to the isomorphism in Proposition 6.2.1 allows us to deduce the next result on determining vanishing ranges.

Proposition 6.3.1. *Let m be the least positive integer such that there exists $\lambda \in X(T)_+$ with $H^m(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) \neq 0$. Then $H^i(G(\mathbb{F}_q), k) \cong H^i(G, \mathcal{G}_r(k)) = 0$ for $0 < i < m$.*

A more detailed analysis is necessary to address (6.1.2) using this aforementioned filtration (cf. [BNP7, Section 2.7]). Our results from this section are summarized below.

Theorem 6.3.2. *Let m be the least positive integer such that there exists $\nu \in X(T)_+$ with $H^m(G, H^0(\nu) \otimes H^0(\nu^*)^{(r)}) \neq 0$. Let $\lambda \in X(T)_+$ be such that $H^m(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) \neq 0$. Suppose $H^{m+1}(G, H^0(\nu) \otimes H^0(\nu^*)^{(r)}) = 0$ for all $\nu < \lambda$ that are linked to λ . Then*

- (i) $H^i(G(\mathbb{F}_q), k) = 0$ for $0 < i < m$;
- (ii) $H^m(G(\mathbb{F}_q), k) \neq 0$;
- (iii) *if, in addition, $H^m(G, H^0(\nu) \otimes H^0(\nu^*)^{(r)}) = 0$ for all $\nu \in X(T)_+$ with $\nu \neq \lambda$, then*

$$H^m(G(\mathbb{F}_q), k) \cong H^m(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}).$$

6.4. Good Filtrations on Cohomology: For $\lambda \in X(T)_+$ an open problem is to determine if $H^i(G, H^0(\lambda))^{(-1)}$ admits a good filtration. When $i = 0, 1, 2$ this holds for all primes [Jan3, BNP4, WJ]. For $p > h$, from [AJ] and [KLT], we have

$$(6.4.1) \quad H^i(G_1, H^0(\nu))^{(-1)} = \begin{cases} \text{ind}_B^G(S^{\frac{i-\ell(w)}{2}}(\mathfrak{u}^*) \otimes \mu) & \text{if } \nu = w \cdot 0 + p\mu \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathfrak{u} = \text{Lie}(U)$. Note also that since $p > h$ and ν is dominant, μ must also be dominant. This explicit description of the cohomology shows that in this case the cohomology groups $H^i(G, H^0(\lambda))^{(-1)}$ admit a good filtration.

For $n > 0$, let $P_n(\nu)$ denote the number of times that ν can be expressed as a sum of exactly n positive roots. Note that $P_0(0) = 1$. The function P_n is often referred to as *Kostant's Partition Function*. By using work of [AJ] and the fact that the cohomology has a good filtration one can show the following result.

Proposition 6.4.1. *Assume $p > h$. Let $\lambda = p\mu + w \cdot 0 \in X(T)_+$. Then*

$$\dim H^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) = \sum_{u \in W} (-1)^{\ell(u)} P_{\frac{i-\ell(w)}{2}}(u \cdot \lambda - \mu).$$

6.5. Upper Bound on Cohomology: Now we can use the existence of the filtration on $\mathcal{G}(k) = k[G/G(\mathbb{F}_q)]$ and the preceding proposition to provide an upper bound on the cohomology group $H^i(G(\mathbb{F}_p), k)$.

Theorem 6.5.1. *Assume $p > h$.*

$$\dim H^i(G(\mathbb{F}_p), k) \leq \sum_{\{w \in W \mid \ell(w) \equiv i \pmod{2}\}} \sum_{\mu \in X(T)_+} \sum_{u \in W} (-1)^{\ell(u)} P_{\frac{i-\ell(w)}{2}}(u \cdot (p\mu + w \cdot 0) - \mu).$$

6.6. We would like to indicate how to address the question of vanishing ranges (6.1.1) by using root combinatorics and Proposition 6.3.1.

Proposition 6.6.1. *Suppose that $\Phi \neq G_2$. Let $p > h$, $\lambda = w \cdot 0 + p\mu$, $\mu \neq 0$ and $H^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$,*

- (a) *For $\sigma \in \Phi^+$, $(p-1)\langle \mu, \sigma^\vee \rangle + \ell(w) + \langle w \cdot 0, \sigma^\vee \rangle \leq i$.*
- (b) *For $\tilde{\alpha}$ the highest root, $(p-1)\langle \mu, \tilde{\alpha}^\vee \rangle - 1 \leq i$.*

Proof. (a): First we have

$$\begin{aligned} 0 \neq H^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) &\cong \text{Hom}_G(V(\lambda)^{(1)}, H^i(G_1, H^0(\lambda))) \\ &= \text{Hom}_G(V(\lambda), \text{ind}_B^G S^{\frac{i-\ell(w)}{2}}(\mathbf{u}^*) \otimes \mu) \\ &= \text{Hom}_B(V(\lambda), S^{\frac{i-\ell(w)}{2}}(\mathbf{u}^*) \otimes \mu). \end{aligned}$$

Observe that $\lambda - \mu$ must be a weight of $S^{\frac{i-\ell(w)}{2}}(\mathbf{u}^*)$. Consequently, $\lambda - \mu$ is expressible as $\frac{i-\ell(w)}{2}$ positive roots.

Next note that for $\sigma_1, \sigma_2 \in \Phi$, $\langle \sigma_1, \sigma_2^\vee \rangle \leq 2$ ($\Phi \neq G_2$). Therefore,

$$\langle \lambda - \mu, \sigma^\vee \rangle \leq \left(\frac{i - \ell(w)}{2} \right) \cdot 2 = i - \ell(w).$$

By substituting $\lambda = p\mu + w \cdot 0$ one has

$$\langle p\mu + w \cdot 0 - \mu, \sigma^\vee \rangle \leq i - \ell(w)$$

and

$$(p-1)\langle \mu, \sigma^\vee \rangle + \ell(w) + \langle w \cdot 0, \sigma^\vee \rangle \leq i.$$

(b) The weight $-w \cdot 0$ can be expressed uniquely as $\ell(w)$ distinct positive roots. Since at most one can be $\tilde{\alpha}$ and $\langle \tilde{\alpha}, \tilde{\alpha}^\vee \rangle = 2$, it follows that

$$\langle -w \cdot 0, \tilde{\alpha}^\vee \rangle \leq (\ell(w) - 1) + 2 = \ell(w) + 1.$$

Thus, $\langle w \cdot 0, \tilde{\alpha}^\vee \rangle \geq -\ell(w) - 1$. Now apply part (a) to obtain the result. \square

6.7. Applications: By applying Proposition 6.3.1 and Theorem 6.3.2 with the LHS spectral sequence we are able to deduce the following theorem which addresses (6.1.1) and (6.1.2) when $\Phi = C_n$ and A_n .

Theorem 6.7.1 (A). *Suppose Φ is of type C_n with $p > 2n$. Then*

- (a) $H^i(G(\mathbb{F}_q), k) = 0$ for $0 < i < r(p-2)$;
- (b) $H^{p-2}(G(\mathbb{F}_p), k) \cong k$;
- (c) $H^{r(p-2)}(G(\mathbb{F}_q), k) \neq 0$.

Theorem 6.7.2 (B). *Suppose Φ is of type A_n with $n \geq 2$. Suppose further that $p > n+1$.*

- (a) *(Generic case) If $p > n+2$ and $n > 3$, then*
 - (i) $H^i(G(\mathbb{F}_p), k) = 0$ for $0 < i < 2p-3$;
 - (ii) $H^{2p-3}(G(\mathbb{F}_p), k) = k$.
- (b) *If $p = n+2$, then*
 - (i) $H^i(G(\mathbb{F}_p), k) = 0$ for $0 < i < p-2$;
 - (ii) $H^{p-2}(G(\mathbb{F}_p), k) = k \oplus k$.
- (c) *If $n = 2$ and 3 divides $p-1$, then*
 - (i) $H^i(G(\mathbb{F}_p), k) = 0$ for $0 < i < 2p-6$;
 - (ii) $H^{2p-6}(G(\mathbb{F}_p), k) = k \oplus k$.
- (d) *If $n = 2$ and 3 does not divide $p-1$, then*
 - (i) $H^i(G(\mathbb{F}_p), k) = 0$ for $0 < i < 2p-3$;
 - (ii) $H^{2p-3}(G(\mathbb{F}_p), k) = k$.

- (e) If $n = 3$ and $p > 5$, then
- (i) $H^i(G(\mathbb{F}_p), k) = 0$ for $0 < i < 2p - 6$;
 - (ii) $H^{2p-6}(G(\mathbb{F}_p), k) = k$.

For the type C_n case when $r = 1$, we have seen that for $p > h = 2n$, $H^i(G(\mathbb{F}_q), k) = 0$ for $0 < i < (p - 2)$. We will now outline how to construct λ such that $H^{p-2}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$. Let $w = s_1 s_2 \cdots s_{n-1} s_n s_{n-1} \cdots s_1 \in W$. Then

$$-w \cdot 0 = n\tilde{\alpha} = 2n\omega_1 = \text{sum of all positive roots containing } \alpha_1.$$

Furthermore, $\lambda = p\omega_1 + w \cdot 0 = (p - 2n)\omega_1$, thus $\lambda - \omega_1 = (\frac{p-1}{2} - n)\tilde{\alpha}$.

Next one shows that

$$P_{\frac{p-1}{2}-n}(u \cdot \lambda - \mu) = \begin{cases} 1 & \mu = 1 \\ 0 & \mu \neq 1. \end{cases}$$

This implies $H^{p-2}(G, H^0(\lambda) \otimes H^0(\lambda^*)) \neq 0$. Finally, one has to also prove that $\lambda \in X(T)_+$ (as above) is the only such weight in $X(T)_+$ such that $H^{p-2}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$ (see [BNP8, Section 5.3]).

We remark that (6.1.1) and (6.1.2) for other types (when G is simply connected) is addressed further in [BNP8]. Moreover, we exhibit uniform bounds when G is of adjoint type.

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